A NOTE ON THE LIMITING DISTRIBUTION OF "JACKSON'S ESTIMATE" OF AVERAGE MORTALITY RATE

BU-130-M D. S. Robson January 24, 1961

ABSTRACT

If n_x is the number of x's in a sample of size n from the distribution

$$f(x)=s_1 s_2 \cdots s_x/(1+s_1+s_1 s_2+\cdots+s_1 s_2 \cdots s_k) \qquad x=0,1,\cdots,k$$

then the maximum likelihood estimator of s_i is $S_i=n_i/n_{i-1}$. The asymptotic distribution of the S vector is the multivariate normal with mean s and a covariance matrix estimated by

$$\hat{\sigma}_{ii}=\frac{n_i^2}{n_{i-1}^2(n_0+n_1+\cdots+n_{i-1})}\left[\frac{n_0+\cdots+n_i}{n_i}+\frac{n_0+\cdots+n_{i-1}}{n_{i-1}}\right]$$

$$\hat{\sigma}_{i,i+1}=-\frac{n_{i+1}}{n_{i-1}n_i}$$

All other covariances are zero in this asymptotic distribution.

The variance of the asymptotic normal distribution of the estimate of the average $\bar{S}=(S_1+\cdots+S_k)/k$ is readily obtained from these results.

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Let X be a chance variable having the probability distribution

$$f(x)=s_1 s_2 \cdots s_x / (1+s_1+s_1 s_2+\cdots+s_1 s_2 \cdots s_k)$$

for $x=0,1,\cdots,k$ and suppose $X_1,\cdots,X_n$ are independent and distributed as X. Let

$$n_x = \text{number } X_i \text{ which are equal to } x$$

then the log likelihood of $X_1,\cdots,X_n$ may be written

$$\log L = (n_1+\cdots+n_k)\log s_1+(n_2+\cdots+n_k)\log s_2+\cdots+n_k \log s_k$$
$$-n \log(1+s_1+s_1 s_2+\cdots+s_1 s_2 \cdots s_k)$$

The maximum likelihood equations then become

$$\frac{\partial \log L}{\partial s_i} = \frac{n_i+\cdots+n_k}{s_i} - \frac{n(s_1\cdots s_i+s_1\cdots s_{i+1}+\cdots+s_1\cdots s_k)}{s_i(1+s_1+s_1 s_2+\cdots+s_1 s_2\cdots s_k)} = 0$$

or, letting

$$\alpha_i=(s_1\cdots s_i+s_1\cdots s_{i+1}+\cdots+s_1\cdots s_k)/(1+s_1+s_1 s_2+\cdots+s_1\cdots s_k)$$

then

$$\frac{\partial \log L}{\partial s_i} = \frac{n_i+\cdots+n_k}{s_i} - \frac{n\alpha_i}{s_i} = 0$$

Consequently, the maximum likelihood estimator of α_i is $A_i=(n_i+\cdots+n_k)/n$ and since

$$s_i = \frac{\alpha_i-\alpha_{i+1}}{\alpha_{i-1}-\alpha_i}$$

then the maximum likelihood estimator of s_i is

$$S_i = \frac{A_i-A_{i+1}}{A_{i-1}-A_i} = \frac{n_i}{n_{i-1}}$$

as was mentioned by Chapman and Robson [1].

The average of the k estimators $S_1,\cdots,S_k$ is referred to as Jackson's estimate [2] and is widely used in fisheries work as an estimator of average annual mortality rate. We shall now compute the covariance matrix of the limiting distribution of $S_1,\cdots,S_k$ and from this obtain the variance of the limiting distribution of

$$\bar{S} = \frac{1}{k}(S_1+\cdots+S_k)$$

The inverse elements of the asymptotic covariance matrix are

$$\sigma^{ii} = -E\,\frac{\partial \log L^2}{\partial s_i^2} = E\left(\frac{n_i+\cdots+n_k}{s_i^2}\right) - \frac{n\alpha_i^2}{s_i^2} = \frac{n\alpha_i(1-\alpha_i)}{s_i^2}$$

and, for $i<j$,

$$\sigma^{ij} = -E\,\frac{\partial \log L^2}{\partial s_i \partial s_j} = \frac{n\alpha_j(1-\alpha_i)}{s_i s_j}$$

This matrix $\|\sigma^{ij}\|$ is of the same form considered in an earlier note [3], and applying those results directly to this problem we obtain the inverted matrix

$$\sigma_{ii} = \frac{s_i^2}{n(1-\alpha_i)}\left[\frac{1-\alpha_{i+1}}{\alpha_i-\alpha_{i+1}} + \frac{1-\alpha_{i-1}}{\alpha_{i-1}-\alpha_i}\right] \quad \text{for } i=1,\cdots,k$$

$$\sigma_{i,i+1} = -\frac{s_i s_{i+1}}{n(\alpha_i-\alpha_{i+1})} \qquad \text{for } i=1,\cdots,k-1$$

and all other covariances are zero. The variance of Jackson's estimate is therefore asymptotically equal to

$$\mathrm{var}(\bar{S}) = \frac{1}{k^2}\left[\sum_{i=1}^{k}\sigma_{ii} + 2\sum_{i=1}^{k-1}\sigma_{i,i+1}\right]$$

Maximum likelihood estimators of these variances and covariances are given by

$$\hat{\sigma}_{ii} = \frac{n_i^2}{n_{i-1}^2(n_0+n_1+\cdots+n_{i-1})}\left[\frac{n_0+\cdots+n_i}{n_i} + \frac{n_0+\cdots+n_{i-1}}{n_{i-1}}\right]$$

$$\hat{\sigma}_{i,i+1} = -\frac{n_{i+1}}{n_{i-1}n_i}$$

References

[1] Chapman, D. G. and D. S. Robson. The analysis of a catch curve. Biometrics Vol. 16:354-68, 1960.

[2] Jackson, C. H. N. The analysis of an animal population. J. Animal Ecology 8:238-46, 1939.

[3] Robson, D. S. Maximum likelihood estimation of a sequence of annual survival rates from a capture-recapture series. Biometrics Unit Mimeograph Series BU-116-M, Cornell University, Ithaca, New York. 1960.